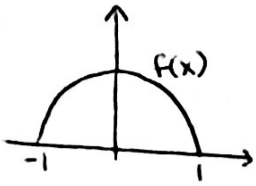
MAT 1322

Jan. 25 Lecture

Fink

Recall: arc length



semi-circle of radius 1 -has length π ($C = \pi r^2$)

$f(x) = \sqrt{1-x^2}$

formula: $L = \int_{-1}^{1} \sqrt{1+f'(x)^2}\,dx$ $\qquad f'(x) = \frac{-x}{\sqrt{1-x^2}}$

$= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$ $\qquad x = \sin(u)$, $dx = \cos(u)\,du$

$= \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{1-\sin^2 u}} \cdot \cos(u)\,du$

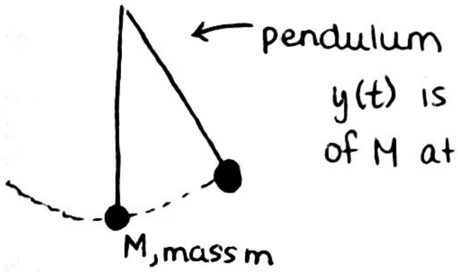
$= \int_{-\pi/2}^{\pi/2} \frac{1}{\cancel{\cos u}} \cdot \cancel{\cos u}\,du$

$= \int_{-\pi/2}^{\pi/2} du$

$= u \Big]_{-\pi/2}^{\pi/2}$

$= \pi$

Differential equations



$y(t)$ is the position of M at the time t

assume m = 1 for this example

$y''(t) \cdot m = -k \cdot y(t)$

★ k is a coefficient specific to the pendulum

$y(t) = \sin(t \cdot \sqrt{k})$ fits

because: $y'(t) = \cos(t\sqrt{k})\sqrt{k}$

$y''(t) = -\sin(t\sqrt{k})\sqrt{k}\sqrt{k}$

put in equation:

$-\sin(t\sqrt{k}) \cdot k = -k\sin(t\sqrt{k})$.

SO, $y(t) = -\sin(t\sqrt{k})k$ is a solution of the differential equation, ie. it satisfies the equation.

Population model

$P(t)$ denotes the population at time t.

Given model: $\frac{dP}{dt} = k \cdot P$ $\qquad (P' = k \cdot P)$

It turns out $P(t) = e^{k \cdot t}$ is a solution

- depending on the sign of k, the population grows or declines

What about $P_2(t) = 5e^{kt}$?

$P_2'(t) = 5e^{kt} \cdot k$

put in equation: $5e^{kt}k = k \cdot 5e^{kt}$

P_2 is also a solution! In general, there are ∞ solutions.

Ex Find the solution of $\frac{dy}{dx} = -5x$

Method of separating variables

$dy = -5x\,dx$ $\qquad$ * multiply equation by dx

$\int dy = \int -5x\,dx$ $\qquad$ * integrate equation

$y + C = -5\frac{x^2}{2} + K$

$y = -5\frac{x^2}{2} + (K - C)$

$y = \frac{-5}{2}x^2 + Z$

Check that $y(x) = \frac{-5}{2}x^2 + Z$ is a solution to $\boxed{y' = -5x}$

$y'(x) = \frac{-5}{2}x \cdot 2 + 0 = -5x$

Ex $\frac{dy}{dx} = \frac{x^2}{y^2}$

$y^2 dy = x^2 dx$

$\int y^2 dy = \int x^2 dx$

$\frac{y^3}{3} = \frac{x^3}{3} + C$

$y^3 = x^3 + 3C$

$y = \sqrt[3]{x^3 + 3C}$ $\qquad C \in \mathbb{R}$

For any choice of C we get a slightly different function as a solution. So we get a family of functions as a solution to a differential equation. How do we know the specific function we want?

→ we observe (measure, count) data (function value) at a starting time t_0. This gives the initial value $y(t_0)$

<u>Ex</u> Find y where $y' = 7x^3$ and and $y(0) = 3$

① Solve equation

② Input initial value

$$\frac{dy}{dx} = 7x^3$$

$$\int dy = \int 7x^3\,dx$$

$$y = \frac{7}{4}x^4 + C.$$

$$\underline{3 = C}$$

$y(x) = \frac{7x^4}{4} + 3$ is the solution.

Back to population model:

$$\frac{dy}{dt} = k \cdot y$$

✡ solve with method of separating variables

coefficient, a constant

$$\int \frac{dy}{y} = \int k\,dt$$

$$\ln|y| = kt + C$$

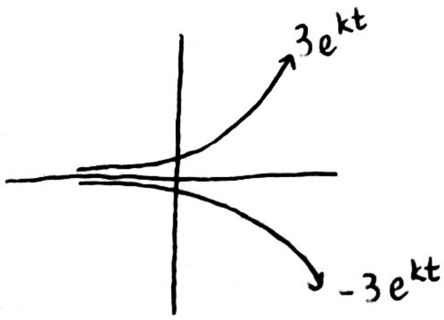
$y = \pm e^{kt+C} = \pm e^{kt} \cdot e^{C} = \pm A e^{kt}$, $A = e^{C}$, a real constant.

What happens with different initial values?

a) $y(0) = 3$
b) $y(0) = -3$

eg. a) $y(0) = \pm A \cdot e^0 = \pm A = 3 \longrightarrow y(t) = 3e^{kt}$

b) $y(0) = A = -3 \longrightarrow y(t) = -3e^{kt}$



"different members of the same family"